Solutions for Exercise Sheet 6

December 26, 2012

1. Use Weierstrass M-test to show that $\sum_{n=1}^{\infty} f_n$ is uniformly convergent. Now it is possible to switch integral and sum to show that integrals around closed curves in U of the function $\sum_{n=1}^{\infty} f_n$ are zero. By Morera's Theorem it is holomorphic in U.

2. Using Fubini's theorem to switch the order of integration and show that integrals around closed curves in $Re(z) > 0$ of the function Γ are zero. By Morera's Theorem it is holomorphic in $Re(z) > 0$.

3. Use the Maximum Modulus for $\exp(f(z))$.

4. Assume in contradiction that $f(z_0) \neq 0$. Use the Maximum Modulus for $\frac{1}{f}$ to obtain a contradiction.

5. (a)
$$\sum_{n=0}^{\infty} -z^n + \sum_{n=0}^{\infty} -(\frac{1}{2})^{n+1} z^n$$
   (b)
$$\sum_{n=0}^{\infty} z^{-n-1} + \sum_{n=0}^{\infty} -(\frac{1}{2})^{n+1} z^n$$
   (c)
$$\sum_{n=0}^{\infty} z^{-n-1} + \sum_{n=0}^{\infty} 2^n z^{-n-1}$$

6. (a)
$$\frac{1}{z^2 - 5z + 6} = -\frac{1}{z-2} - \sum_{n=0}^{\infty} (z-2)^n$$
   (b)
$$(z^2 + 1)\exp(\frac{1}{z}) = z^2 + z + \sum_{n=0}^{\infty} (\frac{1}{n!} + \frac{1}{(n+2)!}) \frac{1}{z^n}$$

(c)

$$\frac{\sin z}{z-2\pi} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(-1)^n (z-2\pi)^{2n}$$

.

(d)

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{\frac{1}{2i}}{z-i} - \frac{\frac{1}{2i}}{z+i}$$

$$= \frac{1}{2i}\left(\frac{1}{(z-1-i)+1} - \frac{1}{(z-1-i)+1+2i}\right)$$

$$= \frac{1}{2i}\left(\frac{\frac{1}{z-1-i}}{1+\frac{1}{z-1-i}} - \frac{\frac{1}{1+2i}}{\frac{(z-1-i)}{1+2i}+1}\right)$$

$$= \frac{1}{2i}\left(\sum_{n=0}^{\infty}(-1)^n (z-(1+i))^{-n-1} - \sum_{n=0}^{\infty}\left(\frac{1}{1+2i}\right)^{n+1}(-1)^n (z-(1+i))^n\right)$$

(e)

$$\frac{1}{1-z} = -\sum_{n=1}^{\infty} z^{-n}$$

7. (a) Look at the Laurent series of $f(z)$. The lowest power of z in the series is $-n$. You can obtain a Laurent series for $f(z^2)$ by substituting z with z^2 in the Laurent series of $f(z)$ and then the lowest power of z in the series is $-2n$.

(b) It is a removable singularity. f and g both have a zero of order 3 at z_0.We may write:

$$f(z) = (z-z_0)^3 \cdot h(z)$$

and:

$$g(z) = (z-z_0)^3 \cdot s(z)$$

When h and s are holomorphic at a neighberhood of z_0 and do not equal zero at z_0.

$$\frac{f}{g} = \frac{h}{s}$$

Which shows that the singularity at z_0 is removable.

(c) It is a pole of order 4. Develop both f and g to a Laurent series around z_0. Obtain the Luarent series of $f \cdot g$ at z_0 as a product. The lowest power of $(z - z_0)$ in the series is -4.

8. (a) $\exp(\frac{1}{z-1})$ has an essential singularity at 1.

   (b) $\frac{\sin z}{z^3}$ has a pole of order 2 at 0.

   (c) $\frac{z^3}{\sin(z^2)}$ has a removable singularity at 0. It is a zero of order 1. The function also has poles of order 1 at $\sqrt{2\pi k}$, at $-\sqrt{2\pi k}$, at $i\sqrt{2\pi k}$ and at $-i\sqrt{2\pi k}$ for $k = 1, 2, 3, \ldots$.

   (d) $\frac{\sin(iz)}{(\exp(z)-1)^2}$ has poles of order 1 at $2\pi ki$ for $k \in \mathbb{Z}$.

   (e) $\exp(\exp(\frac{1}{z}))$ has an essential singularity at 0.